MATH 8210: Exam 2 Study Sheet

Spaces Introduction

linear space: A set X is a linear space (vector space) over a scalar field $\mathbb{K}$ if there exist two algebraic operations addition, $+ : X \times X \to X, (x, y) \to x + y$ and scalar multiplication, $\cdot : \mathbb{K} \times X \to X, (a, x) \to a \cdot x$ such that they satisfy: $\forall x, y, z \in X, \forall a, b \in \mathbb{K}$

1. commutativity: $x + y = y + x$
2. associativity: $(x + y) + z = x + (y + z)$
3. zero element: $\exists \mathbf{0} \in X$ s.t. $x + \mathbf{0} = \mathbf{0} + x = x$
4. inverse element: $\forall x \in X, \exists x \in X$ s.t. $x+(-x) = (-x) + x = \mathbf{0}$
5. compatibility: $a \cdot (b \cdot x) = (ab) \cdot x$
6. multiplicative identity: $1 \cdot x = x$
7. distribution: $(a + b) \cdot x = ax + bx$
8. distribution: $a \cdot (x + y) = a \cdot x + a \cdot y$

linear subspace: A subset $Y \subset X$ is a linear subspace of X, denoted by $Y \leq X$ if $\forall a_1, a_2 \in \mathbb{K}, y_1, y_2 \in Y$, $a_1y_1 + a_2y_2 \in Y$.
span: The span of a subset $A \subset X$, denoted as $Span(A)$ or $\langle A \rangle$ is $Span(A) = \{\sum_{i=1}^n a_i x_i | a_i \in \mathbb{K}, x_i \in A, i = 1, \cdots, n, n \in \mathbb{N}\}$
linearly independent: A subset $A \subset X$ is linearly independent if $\sum_{i=1}^n a_i x_i = \mathbf{0}$, $\forall a_i \in \mathbb{K}, x_i \in A \implies a_i = 0, i = 1, 2, \cdots, n$
Hamel Basis: A subset $A \subset X$ is a Hamel Basis of X if A is linearly independent and $Span(A) = X$.
dimension: The dimension of X is the number of elements in a Hamel Basis of X.
Theorem: Every nonzero vector space has a Hamel Basis and all Hamel bases of X have the same number of elements.
metric linear space (MLS): $(X, d, +, \cdot)$ is called a MLS if

- (X, d) is a metric space
- $(X, +, \cdot)$ is a linear space (over $\mathbb{R}$)
- $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ are continuous

Spaces Introduction Continued

translation-scaling-invarient metric linear space (TSI-MLS): $(X, d, +\cdot)$ is a TSI-MLS if

- (X, d) is a metric space
- $(X, +\cdot)$ is a linear space
- $d(x + z, y + z) = d(x, y)$ and $d(a \cdot x, a \cdot y) = |a|d(x, y), \forall x, y, z \in X, a \in \mathbb{R}$

normed linear space (NLS): $(X, ||\cdot||, +, \cdot)$ is a NLS if $(X, +\cdot)$ is a linear space and there exists a norm $|| \cdot || : X \to \mathbb{R}$ s.t.

- $||x|| \geq 0, ||x|| = 0 \iff x = \mathbf{0}$
  AKA: $||x|| = 0 \implies x = \mathbf{0}$
- $||a \cdot x|| = |a|||x||, \forall a \in \mathbb{R}, x \in X$
- $||x + y|| \leq ||x|| + ||y||, \forall x, y \in X$

Theorem: TSI-MLS $\implies$ MLS
Theorem: NLS = TSI-MLS
Banach Space: A Banach Space is a complete NLS.

Sequences, Series, Schauder Basis

convergent: $(X, || \cdot ||)$ a NLS. A sequence $\{x_n\} \subset X$ is convergent if $\exists x \in X$ s.t. $||x_n - x|| \to 0, n \to \infty$
Cauchy: $(X, || \cdot ||)$ a NLS. A sequence $\{x_n\} \subset X$ is Cauchy if $||x_n - x_m|| \to 0, n, m \to \infty$
convergent: $(X, || \cdot ||)$ a NLS. An infinite series $\sum_{n=1}^\infty x_n$ is convergent if $\exists x \in X$ if $||\sum_{n=1}^N x_n - x|| \to 0, N \to \infty$. **absolutely convergent:** $(X, || \cdot ||)$ a NLS. $\sum_{n=1}^\infty$ is absolutely convergent if $\sum_{n=1}^\infty ||x_n|| < \infty$.
Theorem: Let $(X, || \cdot ||)$ be a NLS. Then $(X, || \cdot ||)$ is Banach $\iff$ If $\sum_{n=1}^\infty x_n$ is absolutely convergent, then $\sum_{n=1}^\infty x_n$ is convergent.
Schauder Basis: Let $(X, ||\cdot||)$ be a NLS. A sequence $\{e_n\} \subset X, e_n \neq \mathbf{0}$ is a Schauder Basis of $(X, || \cdot ||)$ if $\forall x \in X, \exists$ a unique set of coefficients $\{a_n\} \subset \mathbb{R}$ s.t. $x = \sum_{n=1}^\infty a_n e_n$ (i.e. $||x - \sum_{n=1}^N a_n e_n|| \to 0$ as $N \to \infty$)
Lemma: Schauder Basis is linearly independent
Theorem: If $(X, || \cdot ||)$ has a Schauder Basis, then $(X, || \cdot ||)$ is separable.

Finite Dimensional NLS

Linear Combination Theorem: Let $(X, || \cdot ||)$ be a NLS and $\{x_i\}_{i=1}^n$ be finitely many linearly independent vectors. Then, $\exists c > 0$ s.t. $c\sum_{i=1}^n |a_i|||x_i|| \leq ||\sum_{i=1}^n a_i x_i||, \forall a_i \in \mathbb{R}, i = 1, \cdots, n.$
Remarks:

$$
\begin{aligned}
||\sum_{i=1}^n a_i x_i|| &\leq \sum_{i=1}^n ||a_i x_i|| \\
&\leq \sum_{i=1}^n |a_i|||x_i|| \\
&\leq C \sum_{i=1}^n |a_i| \\
&\text{where } C = \max_{1 \leq i \leq n} ||x_i||
\end{aligned}
$$

$$\tilde{c} \sum_{i=1}^n |a_i| \leq ||\sum_{i=1}^n a_i x_i||$$

$$\text{where } \tilde{c} = c \min_{1 \leq i \leq n} ||x_i|| > 0$$

Theorem: Every finite dimensional NLS is Banach.
stronger: Let X be a vector space and $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms defined on X. $|| \cdot ||_1$ is stronger than $|| \cdot ||_2$ if $\exists M > 0$ s.t. $||x||_2 \leq M||x||_1, \forall x \in X$.
equivalent: $|| \cdot ||_1$ is equivalent to $|| \cdot ||_2$ if $\exists m, M > 0$ s.t. $m||x||_2 \leq ||x||_1 \leq M||x||_2, \forall x \in X$.
Remark: $m, M > 0$ are fixed numbers independent of x.
Lemma: If $|| \cdot ||_1$ is stronger than $|| \cdot ||_2$, then

- The identity map $i : (X, || \cdot ||_1) \to (X, || \cdot ||_2)$ is Lipschitz continuous.
- $\{x_n\} \subset (X, || \cdot ||_1)$ is convergent (Cauchy) $\implies$ $\{x_n\} \subset (X, || \cdot ||_2)$ is also convergent (Cauchy).
- $A \subset (X, || \cdot ||_1)$ is dense $\implies$ $A \subset (X, || \cdot ||_2$ is also dense.
- $A \subset (X, ||\cdot||_2)$ open (closed) $\implies$ $A \subset (X, ||\cdot||_1)$ open (closed)

Theorem: All norms on a finite dimensional vector space are equivalent.

Compactness

Let (X,d) be a metric space and $K \subset X$
open cover: A collection of open sets $\{A_i\}_{i\in I} \subset X$ is open cover of K if $\bigcup_{i\in I} A_i \supset K$
compact: K is compact if every open cover of K has a finite subcover. i.e. $\exists\{A_{i_k}\}_{k=1}^n \subset \{A_i\}$ s.t. $\bigcup_{k=1}^n A_{i_k} \supset K$.
totally bounded: K is totally bounded if K can be covered by finitely many open balls with arbitrary small radius. i.e. $\forall\epsilon > 0, \exists\{x_i\}_{i=1}^n \subset K$ s.t. $\bigcup_{i=1}^n B_\epsilon(x_i) \supset K$.
sequentially compact: K is sequentially compact if every sequence in K has a convergent subsequence. i.e. $\forall\{x_n\} \subset K, \exists x \in K, \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \to x$.
Lemma: (X,d) is a metric space and $K \subset X$. Then K is totally bounded $\iff$ $\forall\{x_n\} \subset K, \{x_n\}$ has a Cauchy subsequence.
Theorem: K is sequentially compact $\iff$ K is totally bounded and complete.
Theorem: (X,d) is a metric space and $K \subset X$. K is compact $\iff$ K is sequentially compact.
Lemma: K is compact $\implies$ K is closed and bounded.
Theorem: $(X, ||\cdot||)$ is a finite dimensional NLS and $K \subset X$. K is compact $\iff$ K is closed and bounded.
Continuity Theorem: Let f be a continuous mapping between (X,d_X) and (Y,d_Y). If $K \subset X$ is compact, then $f(K) \subset Y$ is compact.
Riesz's Lemma: $(X, ||\cdot||)$ is a NLS and $Y < X$ is a proper closed subset of X. Then, $\forall\mathbf{0} \in (0,1), \exists x \in X, ||x|| = 1$ s.t. $d(x,y) = \inf_{y\in Y} ||x-y|| \geq \mathbf{0}$.
Remark: If $dim(Y) < \infty$, then $\exists x \in X, ||x|| = 1$ s.t. $d(x,y) = 1$.
Theorem: $(X, ||\cdot||)$ a NLS. $\overline{B}_1(\mathbf{0})$ is compact $\iff$ $dim(X) < \infty$.

Bounded Linear Operators

linear operator: X, Y are vector spaces. A mapping $T: X \to Y$ is called a linear operator if $D(T) \leq X$ and $\forall x_1, x_2 \in D(T), a_1, a_2 \in \mathbb{R}, T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$
Lemma: If $T: X \to Y$ is a linear operator, then

1. $T(\mathbf{0}) = \mathbf{0}$
2. $Range(T) \leq Y$, $Range(T) = \{T(x) | x \in D(T)\}$
3. $Ker(T) \leq D(T) \leq X$, $Ker(T) = \{x \in D(T) | T(x) = \mathbf{0}\}$
4. $dim(D(T)) = dim(Ker(T)) + dim(Range(T))$
5. $Ker(T) = \{\mathbf{0}\}$ $\iff$ T is one-to-one $\iff$ $\exists T^{-1}: Range(T) \to D(T)$ $\iff$ $dim(D(T)) = dim(Range(T))$

linear operator: $\mathcal{L}(X,Y) = \{T : X \to Y | T \text{ is linear}\}$
Theorem: $\mathcal{L}(x,y)$ is a vector space under $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(aT)(x) = aT(x)$
continuous operator: $C(X,Y) = \{T \in \mathcal{L}(X,Y) | T \text{ is continuous}\}$ where X, Y are NLS
Lemma: $C(X,Y) \leq \mathcal{L}(X,Y)$
bounded operator: Let $X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS and $T: X \to Y$. Then T is a bounded operator if $\exists M > 0$ s.t. $||T(x)||_Y \leq M||x||_X, \forall x \in X$
Lemma: If $T \in \mathcal{L}(X,Y)$, then T is bounded $\iff$ $\forall A \subset X$ bounded, $T(A) \subset Y$ bounded.
bounded operator: $B(X,Y) = \{T \in \mathcal{L}(X,Y) | T \text{ is bounded}\}$
Theorem: $T \in \mathcal{L}(X,Y)$. TFAE:

- T is bounded.
- T is Lipschitz continuous.
- T is continuous.
- T is continuous at a single point $x_0 \in X$.

Corollary: Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS and $T: X \to Y$ be linear and bounded. Then,

1. If $x_n \to x$ in $(X, ||x||_X)$, then $T(x_n) \to T(x)$ in $(Y, ||\cdot||_Y)$.
2. $Ker(T)$ is closed in X.

Bounded Linear Operators Continued

Lemma: Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS and $T \in \mathcal{B}(X,Y)$. Then,

$$\inf\{M > 0 | ||T(x)||_Y \leq M||x||_X, \forall x \in X\}$$

$$= \sup_{x\neq\mathbf{0}} \frac{||T(x)||_Y}{||x||_X}$$

$||\mathbf{T}||$: Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS and $T \in \mathcal{B}(X,Y)$. We define $||\cdot|| : \mathcal{B}(X,Y) \to \mathbb{R}$ by

$$||T|| = \sum -x \neq \mathbf{0} \frac{||T(x)||_Y}{||x||_X} = \sup_{||x||_X=1} ||T(x)||_Y$$

Theorem: $(\mathcal{B}(X,Y), ||\cdot||)$ is a NLS with norm $||\cdot||$ defined as above.
Remark: Therefore, $(\mathcal{C}(X,Y), ||\cdot||)$ is also a NLS with the same norm.
Lemma: We have the following:

1. $||T(x)||_Y \leq ||T|| ||x||_X, \forall x \in X$ if $T \in \mathcal{B}(X,Y)$
2. $||S \circ T|| \leq ||S|| ||T||$ if $S \in \mathcal{B}(Y,Z)$ and $T \in \mathcal{B}(X,Y)$
3. $||T^n|| \leq ||T||^n, \forall n \in N$ if $T \in \mathcal{B}(X,X)$.

Theorem: Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS and $dim(X) < \infty$. Then, $\mathcal{B}(X,Y) = \mathcal{L}(X,Y)$.
Theorem: Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be NLS. Then $\mathcal{B}(X,Y)$ is a Banach space if Y is a Banach space.